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STUDY OF FORMAL TRIANGULAR MATRIX RINGS

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***Abstract*—There has been a continuous study on formal triangular matrix rings. In this present study of formal triangular matrix rings few ring theoretic characteristic properties of formal triangular matrix rings have been studied in detail. Some definitive results are verified on these rings concerning properties such as being respectively left Kasch, right mininjective, clean, potent or a ring of stable rank$\leq n$. The concepts of a strong left Kasch ring and a strong right mininjective ring are introduced and it is determined when the triangular matrix rings are respectively strong left Kasch or strong right mininjective**

I. INTRODUCTION

All the rings considered will be associative rings with identity, all the modules considered will be unital modules. For any ring R the category of right (resp left) R-modules will be denoted by Mod-R(resp R-Mod). Let A and B be two given rings and M a left B right A bimodule. The formal triangular matrix ring $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ has its elements formal matrices $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}$ where $a \in A, b \in B$ and $m \in M$ with addition co-ordinate wise and multiplication given by $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \cdot \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ ma' + bm' & bb' \end{bmatrix}$. The present is devoted to study of properties of Formal Triangular rings. Specifically we characterize all maximal (resp minimal) one sided ideals of T, use this to characterize the jacobson radical $J(T)$ and the right (resp left) socle socleT_T (resp soc$_T T$). We determine necessary and sufficient conditions for T to be semi-primary, left(or right) perfect, semi local etc.

II. RING THEORETIC PROPERTIES OF T

(1) Left ideals of T are of the form $I_1 \oplus I_2$ where
$I_1 <_A A, \quad I_2 \leq_B (M \oplus B)$ and $MI_1 \subseteq I_2$; that is $MI_1 \oplus 0 \subseteq I_2$.
(2) Right ideals of T are of the form $J_1 \oplus J_2$ where
$J_2 < B_B,\ J_1 \leqq (A \oplus M)_A$ and $J_2 M \subseteq J_1$.[5]

2. Generalities on formal triangular matrix rings

Proposition II.1. *(1) The set of maximal right ideals of T is given by*
$\{(I \oplus M) \oplus K|$ *either* $I = A$ *and* K *is a maximal right ideal of* B
or I *is a maximal right ideal of* A *and* $K = B\}$
(2) The set of minimal right ideals of T is the union of the two sets,
$\{W \oplus 0| \ W$ *a simple submodule of* $(A \oplus M)_A\}$ *and*
$\{0 \oplus K|$ *with* 0 *the zero submodule of* $(A \oplus M)_A$ *and* K *a minimal right ideal of* B *satisfying* $KM = 0\}$.

Proof. (1) Let $W \oplus K$ with $W \leq (A \oplus M)_A,\ K \leq B_B$ and $KM \leq W$ be a maximal right ideal of T. If $K \neq B$ then choosing a maximal right ideal K' of B_B such that $K \leq K'$, we see that $(A \oplus M) \oplus K'$ is a maximal right ideal of T, but since $W \oplus K$ is maximal we get $W \oplus K = (A \oplus M) \oplus K'$, then this implies that $W = A \oplus M$ and $K = K'$; that is $I = A$ and K is a maximal right ideal of B. On the other hand if $K = B$ then $0 \oplus KM \leq W$ implies that $0 \oplus BM \leq W$; that is $W = I \oplus M$ for some $I \leq A_A$. The maximality of $W \oplus K$ implies that I is a maximal right ideal of A. Thus any maximal right ideal of T has to be either $(A \oplus M) \oplus K$ with K a maximal right ideal of B or $(I \oplus M) \oplus B$ with I a maximal right ideal of A.
(2) Let $W \oplus K$ with $W \leq (A \oplus M)_A,\ K \leq B_B$ and $KM \leq W$ be the minimal right ideal of T. For any $V \leq W$, $V \oplus 0$ is a right ideal of T, maximality of $W \oplus K$ implies that either $K = 0$ and W is a simple submodule of $(A \oplus M)_A$ or K is a minimal right ideal of B_B satisfying $KM = 0$. Conversely, it is clear that $W \oplus 0$ for any simple module and $0 \oplus K$ for any minimal right ideal of B with $KM = 0$ constitute minimal right ideal of T. ∎

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Corollary II.1. *[1] (a)* $J(T) = \begin{bmatrix} J(A) & 0 \\ M & J(B) \end{bmatrix}$ *and*

(b) $soc(T_T) = \begin{bmatrix} soc(A_A) & 0 \\ soc(M_A) & soc(L_B) \end{bmatrix}$ *where* $L = l_B(M)$

Proof. $T = A \oplus M \oplus B$
(a). According to the definition

$J(R)$ = Intersection of maximal right ideals of R. Now maximal right ideals of T are
$\{I \oplus M \oplus B |\ I$ maximal right ideal of A$\} \cup$
$\{A \oplus M \oplus K |\ K$ maximal right ideal of B$\}$ so
$J(T) = J(A) \oplus M \oplus J(B)$
$= \begin{bmatrix} J(A) & 0 \\ M & J(B) \end{bmatrix}$.

(b) According to the definition, for any ring R

soc(R_R) = sum of all minimal right ideals of R. For $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$, by Prop II.1 the minimal right ideals of T are the union of two sets $\{W \oplus 0 |\ W$ a simple submodule of $(A \oplus M)_A\} \cup$ $\{0 \oplus K |\ K$ is a minimal right ideal of B satisfying $KM = 0\}$. Now soc$(T_T) = \sum$ all minimal right ideals of $T = \sum((W \oplus 0) + (0 \oplus K))$ with W and K as above and $KM = 0$.

Since K is minimal right ideal of B such that $KM = 0$ which implies that $K \subseteq l_B(M)$. Therefore
soc(T_T) = soc$(A_A) \oplus$ soc$(M_A) \oplus$ soc$(l_B(M))$

$= \begin{bmatrix} \text{soc}(A_A) & 0 \\ \text{soc}(M_A) & \text{soc}(L_B) \end{bmatrix}$
where $L = l_B(M)$.

□

Corollary II.2. *(i)* $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} + J(T) \rightsquigarrow (a+J(A), b+J(B))$ *is a ring isomorphism of* $T/J(T)$ *with* $A/J(A) \times B/J(B)$.
(ii) Idempotents mod $J(T)$ *can be lifted in* T *if and only if idempotents mod* $J(A)$ *can be lifted in* A *and idempotents mod* $J(B)$ *can be lifted in* B .

Proof. (i) Define $f : T/J(T) \to (A/J(A)) \times (B/J(B))$ by
$f\left(\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} + J(T)\right) = (a + J(A), b + J(B))$
and $g : (A/J(A)) \times (B/J(B)) \to T/J(T)$ by
$g(a + J(A), b + J(B)) = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix}$.

Now $f\left(\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} + J(T)\right) = (J(A), J(B))$ this implies that

$(a + J(A), b + J(B)) = (J(A), J(B))$ so $a \in J(A),\ b \in J(B)$ and $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in \begin{bmatrix} J(A) & 0 \\ M & J(B) \end{bmatrix} = J(T)$. So f is injective.

For any $(a + J(A), b + J(B))$ there exists $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in T/J(T)$ such that $f\left(\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\right) = (A + J(A), b + J(B))$, and so f is surjective.
(ii) [**Lifting Idempotents:** If I is a non empty subset of R then an idempotent of R/I or an idempotent modulo I is an element $x \in R$ such that $x^2 - x \in I$. Such an element is said to be lifted provided that there exists an idempotent $y \in R$ such that $y - x \in I$ and we say that x has been lifted to y.]
Let $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in T$ be an idempotent mod $J(T)$ which can be lifted in T. This implies that $\left(\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\right)^2 - \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = \begin{bmatrix} a^2 - a & 0 \\ ma + bm - m & b^2 - b \end{bmatrix}$
which belongs to $J(T)$; therefore $a^2 - a \in J(A), b^2 - b \in J(B)$; that is a is an idempotent mod $J(A)$ and b is an idempotent mod $J(B)$. And $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}$ can be lifted in T implies that there exists $\begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} \in T$ such that $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} - \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} = \begin{bmatrix} a - a' & 0 \\ m - m' & b - b' \end{bmatrix} \in J(T)$; that is $a - a' \in J(A)$ and $b - b' \in J(B)$, so a is lifted to a' and b is lifted to b'.
Conversely, if a and b are idempotents mod $J(A)$ and $J(B)$ which can be lifted in A and B respectively to a' and b'. Then $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}$ is an idempotent mod $J(T)$ for any m and it can be lifted in T to $\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$. □

Corollary II.3. *T is semilocal if and only if A and B are semilocal. [12]*

Proof. Suppose T is semilocal. This implies that every ideal of $T/J(T)$ is a direct summand of $T/J(T)$. Now by Corollary II.2(i) we have
$T/J(T) \cong (A/J(A)) \times (B/J(B)) \cong (A/J(A)) \oplus (B/J(B))$

So every ideal of $(A/J(A)) \times (B/J(B))$ is a direct summand of $T/J(T)$. The ideals of $(A/J(A)) \times (B/J(B))$ are of the form $\bar{I} \times \bar{J}$ where I and J are ideals of A and B respectively. So each $\bar{I} \times \bar{J}$ is a direct summand of $T/J(T)$; that is $\bar{I}$ is a direct summand of $T/J(T)$ and $\bar{J}$ is a direct summand of $T/J(T)$ hence $\bar{I}$ is a direct summand of $A/J(A)$ and $\bar{J}$ is a direct summand of

$B/J(B)$. Therefore we get $A/J(A)$ and $B/J(B)$ are semsimple and so A and B are semilocal. Proof for the converse is on the similar lines. □

Corollary II.4. *T is semi-perfect if and only if A and B are semi-perfect.*

Proof. By Corollary II.3 we get $T/J(T)$ is semilocal if and only if $A/J(A)$ and $B/J(B)$ are semilocal. Also from corollary II.2(ii) we get that idempotents mod $J(T)$ can be lifted in T if and only if idempotents mod $J(A)$ can be lifted in A and idempotents mod $J(B)$ can be lifted in B. So T is semi-perfect if and only if A and B are semi-perfect. □

Corollary II.5. *T is right (resp left) perfect if and only if A and B are right(resp left) perfect.*

Proof. Proof is clear by Corollary II.3. □

Corollary II.6. *T is semi-primary if and only if A and B are semi-primary.*

Proof. Assume that A and B are semi-primary. Because of corollary II.3, to show that T is semi-primary we need only to show that $J(T)$ is nilpotent. Since $J(A)$ and $J(B)$ are nilpotent we can find an integer $k \geq 1$ with $(J(A))^k = 0$ and $(J(B))^k = 0$.

If $\bar{I}_1 = \begin{bmatrix} J(A) & 0 \\ M & 0 \end{bmatrix}$ and $\bar{I}_2 = \begin{bmatrix} 0 & 0 \\ M & J(B) \end{bmatrix}$, then $\bar{I}_1$ and $\bar{I}_2$ are two sided ideals of T. Now since ${\bar{I}_2}^{k+1} = \begin{bmatrix} (J(A))^{k+1} & 0 \\ M(J(A))^k & 0 \end{bmatrix}$, It is easy to see that ${\bar{I}_1}^{k+1} = 0 = {\bar{I}_2}^{k+1}$. From $J(T) = \bar{I}_1 + \bar{I}_2$, we immediately conclude that $J(T)$ is nilpotent.

Now to prove the converse; again because of Corollary II.3 we have only to show that $J(T)$ is nilpotent implies that $J(A)$ and $J(B)$ are nilpotent.

$$J(T)^k = \begin{bmatrix} (J(A))^k & 0 \\ M(J(A))^{k-1} + (J(B))^{k-1}M & (J(B))^k \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ □

We will now give an exact description of the principal right ideal uT of T where $u = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix}$ and obtain necessary and sufficient conditions for uT to be a minimal right ideal of T.

Lemma II.1. *Let $u = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in T$. Writing $\begin{bmatrix} a \\ m \end{bmatrix} A$ for $\{\begin{bmatrix} a\lambda & 0 \\ m\lambda & 0 \end{bmatrix} \mid \lambda \in A\}$, we have $uT = \begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix}$.*

Proof. For any $\begin{bmatrix} \lambda & 0 \\ n & \mu \end{bmatrix} \in T$; we have

$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ n & \mu \end{bmatrix} = \begin{bmatrix} a\lambda & 0 \\ m\lambda + bn & b\mu \end{bmatrix}$$
$$= \begin{bmatrix} a\lambda & 0 \\ m\lambda & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ bn & b\mu \end{bmatrix}.$$

Hence $uT \subseteq \begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix}$.

From

$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a\lambda & 0 \\ m\lambda & 0 \end{bmatrix}$$
$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ bn & 0 \end{bmatrix}$$

and $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b\mu \end{bmatrix}$; we see that $\begin{bmatrix} a \\ m \end{bmatrix} A \subseteq uT$,

$\begin{bmatrix} 0 & 0 \\ bM & 0 \end{bmatrix} \subseteq uT$ and $\begin{bmatrix} 0 & 0 \\ 0 & bB \end{bmatrix} \subseteq uT$. □

Corollary II.7. *Let $u = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in T$. Then uT is a minimal right ideal of T if and only if one of the following is true.*
(1) $b = 0$ and (a, m) generates a simple submodule of $(A \oplus M)_A$.
(2) $a = 0,\ m = 0,\ bB$ is a minimal right ideal B and $bM = 0$.

Proof. From (2) of Proposition II.1 and Lemma II.1 we see that uT is minimal right ideal of T if and only if one of the following is valid.
(i) $\begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix} = W \oplus 0$ where W is a simple submodule of $(A \oplus M)_A$.
(ii) $\begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix} = 0 \oplus K$ where K is a minimal right ideal of B satisfying $KM = 0$.

Now (i) is valid if and only if $b = 0$ and (a, m) generates a simple submodule of $(A \oplus M)_A$ and (ii) is valid if and only if $a = 0$, $m = 0$ and bB is a minimal right ideal of B and $bM = 0$. □

We take the analogue of Proposition II.1 is valid for left ideals of T. We state it without proof.

Proposition II.2. *(1) The set of maximal left ideals of T is given by*
$\{I \oplus (M \oplus K)|$ either $I = A$ and K is a maximal left ideal of B
or I is a maximal left ideal of A and $K = B\}$.
(2) The set of minimal left ideals of T is the union

of the two sets
$\{0 \oplus V|\ V$ *a simple submodule of* $_B(M \oplus B)\} \cup$
$\{I \oplus 0|$ *with 0 the zero submodule of* $_B(M \oplus B)$ *and* I *a minimal left ideal of* A *satisfying* $MI = 0\}$.

In particular, we see from (2) above that $Soc(_TT) = \begin{bmatrix} Soc(_AH) & 0 \\ Soc(_BM) & Soc(_BB) \end{bmatrix}$, *where* $H = r_A(M)$.

III. Left Kasch Nature, Right Mininjectivity

3. Left (right) Kasch and strong left (strong right) Kasch rings.

Definition III.1. A ring R is said to be left Kasch if $r_R(I) \neq 0$ for every maximal left ideal I of R.

Definition III.2. R will be called a strong left Kasch ring if $r_R(I)$ is a minimal right ideal of R for every maximal left ideal I of R.

This section is devoted to studying conditions under which the triangular matrix ring $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ is left Kasch (resp strong left Kasch).

$$r_R(I) = \{r \in R|\ a.r = 0 \text{ for all a } \in I\}.$$

Lemma III.1. *(1) Let* I *be any left ideal of* A. *For the left ideal* $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$ *we have* $r_T(\tilde{I}) = \begin{bmatrix} r_A(I) \cap r_A(M) & 0 \\ 0 & 0 \end{bmatrix}$.

In particular $r_T(\tilde{I})$ *is a minimal left ideal of* T *if and only if* $r_A(I) \cap r_A(M)$ *is a minimal right ideal of* A.
(2) Let K *be any left ideal of* B. *For the left ideal* $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ *of* T *we have* $r_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ r_M(K) & r_B(K) \end{bmatrix}$. *In this case* $r_T(\tilde{K})$ *is a minimal right ideal of* T *if and only if one of the following is valid: Either*
(a) $r_B(K) = 0$ *and* $r_M(K)$ *is a simple submodule of* M_A. *Or*
(b) $r_B(K)$ *is a minimal right ideal of* B *and* $r_B(K)M = 0 = r_M(K)$.

Proof. (1) $r_T(\tilde{I}) = \{t \in T|\ \tilde{I}.t = 0\}$.
Let $t = \begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix}$. Now $\tilde{I}.t = 0$ implies that $\begin{bmatrix} I & 0 \\ M & B \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = 0$; that is $\begin{bmatrix} Ia_1 & 0 \\ Ma_1 + Bm_1 & Bb_1 \end{bmatrix} = 0$. Now $Ia_1 = 0$ implies that $aa_1 = 0$ for all $a \in I$ and so $a_1 \in r_A(I)$ also $Bb_1 = 0$ gives us $bb_1 = 0$ for all $b \in B$ so $b_1 = 0$. Since Ma_1 is a simple module and $Ma_1 + Bm_1 \subseteq M \subseteq M \oplus B$, so $Ma_1 + Bm_1 = 0$ implies that $Ma_1 = 0$ and $Bm_1 = 0$. Now $Ma_1 = 0$ means $a_1 \in r_A(M)$ and $Bm_1 = 0$ means $bm_1 = 0$ for all $b \in B$ proving $m_1 = 0$. Therefore $t = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$ where $a_1 \in r_A(I) \cap r_A(M)$ and hence $r_T(\tilde{I}) = \begin{bmatrix} r_A(I) \cap r_A(M) & 0 \\ 0 & 0 \end{bmatrix}$.

Also $r_T(\tilde{I})$ is a minimal right ideal of T if and only if $r_A(I) \cap r_A(M)$ is minimal right deal of A.
(2) $r_T(\tilde{K}) = \{t \in T|\tilde{K}t = 0\}$. Let $t = \begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in r_T(\tilde{K})$, then $\tilde{K}.t = 0$; that is $\begin{bmatrix} A & 0 \\ M & K \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = 0$, so $\begin{bmatrix} Aa_1 & 0 \\ Ma_1 + Km_1 & Kb_1 \end{bmatrix} = 0$. Thus we have $Aa_1 = 0$ and $Kb_1 = 0$ which gives $a_1 = 0$ and $b_1 \in r_B(K)$. And also $Ma_1 + Km_1 = 0$ gives $Km_1 = 0$; that is $m_1 \in r_M(K)$. Therefore $t = \begin{bmatrix} 0 & 0 \\ m_1 & b_1 \end{bmatrix}$ where $m_1 \in r_M(K)$ and $b_1 \in r_B(K)$. So $r_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ r_M(K) & r_B(K) \end{bmatrix}$. $r_T(\tilde{K})$ is a minimal right ideal of T if either $r_B(K) = 0$ and $r_M(K)$ is a simple submodule of M_A or $r_B(K)$ is a minimal right ideal of B satisfying $r_B(K)M = 0 = r_M(K)$ by using (2) of Prop II.1 □

Theorem III.1. *(i) If* T *is left Kasch, so is* A. *Further* $r_A(I) \cap r_A(M) \neq 0$ *for every maximal left ideal* I *of* A.
(ii) Suppose T *is left Kasch and* $soc(_BM) = 0$. *Then* B *is left Kasch.*
(iii) Let A *and* B *be left Kasch. Further assume that* $r_A(I) \cap r_A(M) \neq 0$ *for every maximal left ideal* I *of* A. *Then* T *is left Kasch.*

Proof. (i) Assume T is left Kasch. Let I be any maximal ideal of A, then $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$ is a maximal left ideal of T and so $r_T(\tilde{I}) \neq 0$. By using Lemma III.1 we get $r_T(\tilde{I}) = \begin{bmatrix} r_A(I) \cap r_A(M) & 0 \\ 0 & 0 \end{bmatrix} \neq 0$ and hence we get $r_A(I) \cap r_A(M) \neq 0$ for every maximal left ideal I of A.
(ii) Assume that T is left Kasch and let K be any maximal left ideal of B. Then $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal left ideal of T. Hence $r_T(\tilde{K}) =$

$\begin{bmatrix} 0 & 0 \\ r_M(K) & r_B(K) \end{bmatrix} \neq 0$. Since $r_M(K) \subset Soc(_BM)$ we get $r_M(K) = 0$. Hence $r_B(K) \neq 0$, showing that B is left Kasch.
(iii) Assume that A and B are left Kasch and that $r_A(I) \cap r_A(M) \neq 0$ for every maximal left ideal I of A. For any maximal left ideal $\tilde{I}$ of T of the form $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$, the equality $r_T(\tilde{I}) = \begin{bmatrix} r_A(I) \cap r_A(M) & 0 \\ 0 & 0 \end{bmatrix}$ shows that $r_T(\tilde{I}) \neq 0$. For the maximal left ideal $\tilde{K}$ of the form $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ with K any maximal left ideal of B the equality $r_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ r_M(K) & r_B(K) \end{bmatrix}$ shows that $r_T(\tilde{K}) \neq 0$ since $r_B(K) \neq 0$. This proves that T is left Kasch. □

We remark that in case $r_A(M)$ is an essential right ideal of A, from (iii) above we see that A, B left Kasch implies that T is left Kasch.

Theorem III.2. *(i) Assume that A and B are strong left Kasch and that $r_A(M)$ is an essential right ideal of A. Then T is strong left Kasch provided $r_M(K) = r_B(K)M = 0$ for all maximal left ideals K of B.*
(ii) Let B be left Kasch. Assume that either $Soc(_AA) \subseteq r_A(M)$ or that $r_A(M)$ is essential in A_A and $r_A(I) \subseteq Soc(A_A)$ for every maximal left ideal I of A. Then T strong left Kasch implies that A and B are strong left Kasch and further $r_M(K) = r_B(K)M = 0$ for all maximal left ideals K of B.

Proof. (i) Assume that A, B are strong left Kasch, $r_A(M)$ is an essential right ideal of A and $r_M(K) = r_B(K)M = 0$ for all maximal left ideals K of B. When I is a maximal left ideal of A, since A is strong left Kasch $r_A(I)$ is a minimal right ideal of A. Since $r_A(M)$ is essential right ideal of A we get $r_A(M) \cap r_A(I) = r_A(I)$. From Lemma III.1 we see that $r_T(\tilde{I})$ is a minimal right ideal of T where $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$. Also, when K is a maximal left ideal of B, from the strong left Kasch nature of B; we see that $r_B(K)$ is a minimal right ideal of B. From (2) of Lemma III.1 we have $r_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ r_M(K) & r_B(K) \end{bmatrix}$, where $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$. The assumption $r_M(K) = r_B(K)M = 0$ shows that $r_T(\tilde{K})$ is a minimal right ideal of T. Since ideals of the form $\tilde{I}$ or $\tilde{K}$ give all the maximal left ideals of T, we see that T is strong left Kasch.
(ii) Let I be any maximal left ideal of A. Then $\tilde{I}$ is a maximal left ideal of T and $r_T(\tilde{I}) = \begin{bmatrix} r_A(I) \cap r_A(M) & 0 \\ 0 & 0 \end{bmatrix}$, since T is strong left Kasch $r_A(M) \cap r_A(I)$ is a minimal right ideal of A. We have $Soc(_AA) \subseteq r_A(M)$ and $r_A(I) \subseteq Soc(_AA)$ we get $r_A(I) \cap r_A(M) = r_A(I)$. On the other hand if $r_A(M)$ is essential in A_A and $r_A(I) \subseteq Soc(A_A)$ so we get $r_A(I) \cap r_A(M) = r_A(I)$, thus $r_A(I)$ is a minimal right ideal of A. This shows that A is a strong left Kasch ring.

Let K be any maximal left ideal of B. The assumption that B is left kasch implies that $r_B(K) \neq 0$. Since $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal left ideal of T and T is strong left Kasch, it follows that $r_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ r_M(K) & r_B(K) \end{bmatrix}$ is a minimal right ideal of T. Since $r_B(K) \neq 0$ from Prop II.1 we see that $r_B(K)$ is a minimal right ideal of B and $r_M(K) = 0 = r_B(K)M$. □

There is an obvious definition of a right (resp strong right) Kasch ring. A ring R is right (resp strong right) Kasch ring if $l_R(I) \neq 0$ (resp $l_R(I)$ is a minimal left ideal of R) for every maximal right ideal of R. We state the following analogues for Theorems III.1 and III.2.

Theorem III.3. *(i) If T is right Kasch so is B. Further $l_B(K) \cap l_B(M) \neq 0$ for every maximal right ideal K of B.*
(ii) Suppose that T is right Kasch and $Soc(M_A) = 0$. Then A is right Kasch.
(iii) Let A and B be right Kasch. Further assume that $l_B(K) \cap l_B(M) \neq 0$ for every maximal right ideal K of B. Then T is right Kasch.

Proof. (i) let K be any maximal right ideal of B then $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal right ideal of T. Since T is right kasch we have $l_T(\tilde{K}) \neq 0$. Now to find $l_T(\tilde{K})$, let $t = \begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in l_T(\tilde{K})$. This implies that $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} A & 0 \\ M & K \end{bmatrix} = \begin{bmatrix} a_1A & 0 \\ m_1A + b_1M & b_1K \end{bmatrix} = 0$. Now $a_1A = 0$ implies $a_1 = 0$, $b_1K = 0$ implies $b_1 \in l_B(K)$ and $m_1A + b_1M = 0$ implies that $m_1A = 0$ and $b_1M = 0$. So $m_1A = 0$ gives $m_1 = 0$ and $b_1M = 0$ gives $b_1 \in l_B(M)$.

Therefore $l_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(K) \cap l_B(M) \end{bmatrix}$. If T is right Kasch then $l_T(\tilde{K}) \neq 0$ so $l_B(K) \cap l_B(M) \neq 0$ for every maximal right ideal $\tilde{K}$ and so $l_B(K) \neq 0$. Thus B is right Kasch.

(ii) Let I be any maximal right ideal of A, then $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$ is a maximal right ideal of T. Now $l_T(\tilde{I}) = \begin{bmatrix} l_A(I) & 0 \\ l_M(I) & 0 \end{bmatrix}$. Since T is right Kasch $l_T(\tilde{I}) \neq 0$, but we have $l_M(I) \subset Soc(M_A)$ so we get $l_M(I) = 0$ and so $l_A(I) \neq 0$ for every maximal right ideal of A. Thus A is right Kasch.

(iii) Let $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$ be any maximal right ideal of T. Now
$l_T(\tilde{I}) = \begin{bmatrix} l_A(I) & 0 \\ l_M(I) & 0 \end{bmatrix}$. Since A is right Kasch $l_A(I) \neq 0$ and so
$l_T(\tilde{I}) \neq 0$. Also if $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal right ideal of T then
$l_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(K) \cap l_B(M) \end{bmatrix}$, which implies that $l_T(\tilde{K}) \neq 0$ and so T is right Kasch. □

Theorem III.4. *(i) Assume that A and B are strong right Kasch and that $l_B(M)$ is an essential left ideal of B. Then T is strong right Kasch provided $l_M(I) = Ml_A(I) = 0$ for all maximal right ideals I of A.*
(ii) Let A be right Kasch. Assume that either $Soc(B_B) \subseteq l_B(M)$ or that $l_B(M)$ is essential in ${}_BB$ and $l_B(K) \subseteq Soc({}_BB)$ for every maximal right ideal K of B. Then T is strong right Kasch implies that A and B are strong right Kasch and further $l_M(I) = Ml_A(I) = 0$ for all maximal right ideals I of A.

Proof. (i) Assume A, B strong right Kasch and $l_B(M)$ is an essential right ideal of B and $l_M(I) = Ml_A(I) = 0$ for all maximal right ideals I of A. Let $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$ be a maximal right ideal of T then $l_T(\tilde{I}) = \begin{bmatrix} l_A(I) & 0 \\ l_M(I) & 0 \end{bmatrix}$. Since A is strong right Kasch $l_A(I)$ is minimal left ideal of A and given that $l_M(I) = Ml_A(I) = 0$, so by using Prop II.2 we get that $l_T(\tilde{I})$ is minimal left ideal of T. Also if $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal right ideal of T then $l_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(K) \cap l_B(M) \end{bmatrix}$. Since B is strong right Kasch $l_B(K)$ is a minimal left ideal of B, also since $l_B(M)$ is essential left ideal of B we have $l_B(M) \cap l_B(K) = l_B(K)$ and so $l_B(K) \cap l_B(M)$ is a minimal left ideal of B and $l_T(\tilde{K})$ is minimal left ideal of T showing that T is strong right Kasch.

(ii) Let K be any maximal right ideal of B, then $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal right ideal of T. Now $l_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(K) \cap l_B(M) \end{bmatrix}$, since T is strong right Kasch $l_T(\tilde{K})$ is a minimal left ideal of T; that is $l_B(K) \cap l_B(M)$ is minimal left ideal of B. We have $l_B(K) \subseteq Soc(B_B)$ since K is maximal right ideal of B. In case $Soc(B_B) \subseteq l_B(M)$ we get $l_B(K) \cap l_B(M) = l_B(K)$. On the other hand if $l_B(M)$ is essential in ${}_BB$ we get $l_B(K) \subseteq Soc({}_BB) \subseteq l_B(M)$ and so $l_B(M) \cap l_B(K) = l_B(K)$. Thus $l_B(K)$ is a minimal left ideal of B showing that B is strong right Kasch.

Now let $\tilde{I} = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$ be a maximal right ideal of T then
$l_T(\tilde{I}) = \begin{bmatrix} l_A(I) & 0 \\ l_M(I) & 0 \end{bmatrix}$. Since T is strong right Kasch $l_T(\tilde{I})$ is minimal left ideal of T. Also since A is right Kasch $l_A(I) \neq 0$ and $l_T(\tilde{I})$ is minimal left ideal of T implies from Prop II.2 that $l_A(I)$ is minimal right ideal of A and
$l_M(I) = Ml_A(I) = 0$ for all maximal right ideals I of A. □

4. Right mininjective and strong right mininjective rings

A ring R is said to be right mininjective if and only if any right R-homomorphism $f : I \to R$ of simple right ideal I of R into R extends to a homomorphism $g : R \to R$ in Mod-R. Equivalently any such f is of the form $f(x) = rx$ for all $x \in I$ for some fixed $r \in R$. Clearly any right injective ring is right mininjective.[7]

If R is a ring, a right module M_R is called mininjective if for each right ideal K of R, every R-morphism $\gamma : K \to M$ extends to R. Equivalently if $\gamma = m.$ is left multiplication by some element of M. Mininjective left modules are defined similarly. Clearly every injective module is mininjective. [Let Q be an R-module. (**Baer's Criterion**) The module Q is injective if and only if for every left ideal I of R any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$.] Our interest is in the right mininjective rings that is the rings R for which R_R is mininjective.

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Lemma III.2. *The following conditions are equivalent for a ring R:*
(1) R is right mininjective.
(2) If kR is simple, $k \in R$, then $lr(k) = Rk$.
(3) If kR is simple and $r(k) \subseteq r(a)$; $k, a \in R$, then $Ra \subseteq Rk$.
(4) If kR is simple and $\gamma : kR \to R$ is R-linear, then $\gamma(k) \in Rk$.
Where $r(k) = \{b \in R | \ kb = 0\}$ and $lr(k) = \{a \in R | \ ab = 0; \ \forall \ b \in r(k)\}$

Proof. Given (1), Let $0 \neq a \in lr(k)$. Then $\gamma : kR \to R$ is well defined by $kr \mapsto ar$. Since R is right mininjective we get $\gamma = c$. where $c \in R$ and γ extends to homomorphism $\gamma' : R \to R$ then $a = ck$, thus $lr(k) = Rk$. Other implications are routine verifications. □

Duality considerations:
If M_R is a right R-module, define the dual of M as $M^d = Hom_R(M, R_R)$. This is a left R-module, where, if $r \in R$ and $\lambda \in M^d$, the map $r\lambda \in M^d$ is defined by $(r\lambda)(m) = r\lambda(m)$ for all $m \in M$.

Lemma III.3. *If $M = mR$ is a principal module and $T = r(m)$, then*
$M^d \cong l(T) = lr(m)$.

Proof. Let $b \in l(T)$, then the map $\lambda_b : M \to R$ is well defined by
$\lambda_b(mr) = br$. Then $b \mapsto \lambda_b$ is an isomorphism $l(T) \to M^d$ of left R-modules. □

[If $\lambda_{b_1} = \lambda_{b_2}$ means that $\lambda_{b_1}(m) = \lambda_{b_2}(m)$; that is $b_1 = b_2$ and so we get injectivity. Also if $\lambda \in M^d$, $\lambda : M_R \to R$ such that $\lambda(m) = a$, then $\lambda(mr) = ar$. Now $\lambda(m.r(m)) = 0 = a.r(m)$ so $a \in lr(m)$ and we get surjectivity.]

Proposition III.1. *The following are equivalent for a ring R*
(1) R is right mininjective.
(2) M^d is simple or zero for all simple right R-modules M.
(3) $l(T)$ is simple or zero for all maximal right ideals T of R.

Proof. (1) $\Rightarrow$ (2): Let $\gamma, \ \delta \in M^d$, where M_R is simple and assume that $\gamma \neq 0$ then we have $\delta o \gamma^{-1} : \gamma(M) \to R$, so since M_R is simple; $\gamma(M)$ is simple right ideal of R. Since R is right mininjective we have $\delta o \gamma^{-1} = a$. for some $a \in R$. So $\delta = a\gamma$ in M^d and so M^d is simple.
(2) $\Rightarrow$ (3): If T is maximal right ideal of R then $(R/T)_R$ is simple. Let $M = (R/T)_R = (m) = mR$ then $T = r(m)$, by previous Lemma $M^d \cong l(T)$ and so $l(T)$ is either simple or zero for all maximal right ideals T of R.
(3) $\Rightarrow$ (1): Given $l(T)$ is simple or zero for all maximal right ideals T of R. To show that R is right mininjective: Let $\gamma : kR \to R$ be R-linear, where kR is minimal right ideal of R and let $i : kR \to R$ be the inclusion map. If $T = r(k)$ then $r(k)$ is a maximal right ideal of R so $l(T)$ is simple or zero. By Lemma III.3 $l(T) = K^d$ where $K = kR$, so K^d is simple. Thus $\gamma = ci$ in K^d for some $c \in R$. So $\gamma = c$. proving that R is right mininjective. □

In this section we study triangular matrix rings $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ which are right mininjective.
$r_A(a) = \{a' \in A | \ aa' = 0\}$, $l_M r_A(a) = \{n \in M | \ na' = 0; \ \forall \ a' \in r_A(a)\}$,
$l_B(M) = \{b \in B | \ bM = 0\}$, $r_B(K) = \{b \in B | \ Kb = 0\}$.

Theorem III.5. *Consider the following conditions (1),(2),(3)and (4).*
(1) A and B are right mininjective and for $a \in A, m \in M$ with $r_A(a)$
and $r_A(m)$ maximal right ideals of A we have $l_M(r_A(a)) \subseteq Ma$,
$l_A r_A(m) = 0$ and $l_M r_A(m) \subseteq Bm$.
(2) T is right mininjective.
(3) $l_B(M)$ is an essential left ideal of B and $l_B(K) \subseteq Soc(_B B)$ for
every maximal right ideal K of B.
(4) $l_B(M)$ is an essential right ideal of B and $l_B(J(B)) = Soc(B_B)$.
Then we have following implications :(1) $\Rightarrow$ (2),
(2) *and* (3) $\Rightarrow$ (1),
(2) *and* (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): In view of Lemma III.2[9], it suffices to show that for any minimal right ideal uT of T the equality $lr(u) = Tu$ holds. Since $u.r(u) = 0$ we always have $u \in lr(u)$, hence $Tu \subseteq lr(u)$. Therefore it suffices to show that $lr(u) \subseteq Tu$.

Since uT is minimal right ideal of T, from Corollary II.7 we see that one of the following is valid:
(i) $u = \begin{bmatrix} a & 0 \\ m & 0 \end{bmatrix}$ with $(a, m)A$ a simple submodule of $(A \oplus M)_A$ or
(ii) $u = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ with $bM = 0$ and bB is minimal right ideal of B.

Suppose (i) holds, to find $r(u)$: Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in r(u)$, so

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$\begin{bmatrix} a & 0 \\ m & 0 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = 0$ which implies $\begin{bmatrix} aa_1 & 0 \\ ma_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ this gives us $aa_1 = 0, ma_1 = 0$; that is $a_1 \in r_A(a) \cap r_A(m)$ and m_1, b_1 can be anything, writing $I = r_A(a) \cap r_A(m)$ we have $r(u) = \begin{bmatrix} I & 0 \\ M & B \end{bmatrix}$.

Now to find $lr(u)$:Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in lr(u)$. So $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix}\begin{bmatrix} I & 0 \\ M & B \end{bmatrix} = \begin{bmatrix} a_1 I & 0 \\ m_1 I + b_1 M & b_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, this gives
$a_1 \in l_A(I),\ b_1 = 0,\ m_1 I + b_1 M = 0$; that is $m_1 \in l_M(I)$. Therefore
$lr(u) = \begin{bmatrix} l_A(I) & 0 \\ l_M(I) & 0 \end{bmatrix}$ and $Tu = \left\{\begin{bmatrix} \lambda a & 0 \\ na + \mu m & 0 \end{bmatrix} | \lambda \in A, n \in M, \mu \in B\right\}$.
Since $(a, m)A \cong A/r_A(a, m) \cong R/I$ is simple so I must be maximal ideal of A. So $r_A(a) \cap r_A(m)$ is maximal and hence exactly one of the following is valid.
(α) $I = r_A(a)$ and $m = 0$
(β) $I = r_A(m)$ and $a = 0$
(γ) $I = r_A(a) = r_A(m)$.
If (α) holds then $lr(u) = \begin{bmatrix} l_A(r_A(a)) & 0 \\ l_M(r_A(a)) & 0 \end{bmatrix}$ and $Tu = \begin{bmatrix} Aa & 0 \\ Ma & 0 \end{bmatrix}$. Since A is right mininjective, we get $l_A(r_A(a)) \subseteq Aa$ and given $l_M(r_A(a)) \subseteq Ma$ so we get $lr(u) \subseteq Tu$.
If (β) holds then $lr(u) = \begin{bmatrix} l_A r_A(m) & 0 \\ l_M r_A(m) & 0 \end{bmatrix}$ and $Tu = \begin{bmatrix} 0 & 0 \\ Bm & 0 \end{bmatrix}$, now given that $l_A r_A(m) = 0$ and $l_M r_A(m) \subseteq Bm$ so $lr(u) \subseteq Tu$.
If (γ) holds then $lr(u) = \begin{bmatrix} l_A r_A(a) & 0 \\ l_M r_A(m) & 0 \end{bmatrix}$ and $Tu = \begin{bmatrix} Aa & 0 \\ Ma + Bm & 0 \end{bmatrix}$, since A is right mininjective $l_A r_A(a) \subseteq Aa$ and given that $l_M r_A(m) \subseteq Bm$ we get $lr(u) \subseteq Tu$.
In case if (ii) is valid $u = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ with $bM = 0$, bB is minimal right ideal of B. To find $lr(u)$: Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in r(u)$, now $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ bm_1 & bb_1 \end{bmatrix} = 0$ which gives $bm_1 = 0,\ bb_1 = 0$; that is $b_1 \in r_B(b)$. Therefore $r(u) = \begin{bmatrix} A & 0 \\ M & r_B(b) \end{bmatrix}$. Now let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in lr(u)$ so $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix}\begin{bmatrix} A & 0 \\ M & r_B(b) \end{bmatrix} = \begin{bmatrix} a_1 A & 0 \\ m_1 a + b_1 M & b_1 r_B(b) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives us $a_1 = 0$, $b_1 \in l_B(r_B(b))$ and $m_1 A + b_1 M = 0$ gives $m_1 = 0,\ b_1 M = 0$; that is $b_1 \in l_B(M)$ and so $lr(u) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(M) \cap l_B(r_B(b)) \end{bmatrix}$ and $Tu = \begin{bmatrix} 0 & 0 \\ 0 & Bb \end{bmatrix}$. Since B is right mininjective $l_B(r_B(b)) \subseteq Bb$ hence
$lr(u) \subseteq Tu$.
(2) and (3) $\Rightarrow$ (1) Let $a \in A$ satisfy the condition that aA is minimal ideal of A. Then $u = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ generates a minimal right ideal of T; because $(a, 0)A$ is simple submodule of $(A \oplus M)_A$. Since T is right mininjective we get $lr(u) \subseteq Tu$. Now to find $lr(u)$: Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in r(u)$, then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = \begin{bmatrix} aa_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives us $a_1 \in r_A(a),\ m_1 \in M, b_1 \in B$, therefore $r(u) = \begin{bmatrix} r_A(a) & 0 \\ M & B \end{bmatrix}$. Now let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in lr(u)$ so $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix}\begin{bmatrix} r_A(a) & 0 \\ M & B \end{bmatrix} = \begin{bmatrix} a_1 r_A(a) & 0 \\ m_1 r_A(a) + b_1 M & b_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives us $a_1 \in l_A r_A(a)$, $b_1 = 0$ and $m_1 \in l_M r_A(a)$. Therefore $lr(u) = \begin{bmatrix} l_A r_A(a) & 0 \\ l_M r_A(a) & 0 \end{bmatrix}$, $u = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $Tu = \begin{bmatrix} Aa & 0 \\ Ma & 0 \end{bmatrix}$. Since $lr(u) \subseteq Tu$ we get $l_A r_A(a) \subseteq Aa$ and $l_M r_A(a) \subseteq Ma$. Since aA is minimal $\Leftrightarrow r_A(a)$ is maximal right ideal of A, we see that $l_M r_A(a) \subseteq Ma$ whenever $r_A(a)$ is a maximal right ideal of A. From $l_A r_A(a) \subseteq Aa$ we conclude that A is right mininjective.

Now let $m \in M$ satisfy the condition that $r_A(m)$ is maximal right ideal of A. Then $(0, m)A \cong A/r_A(m)$ is simple submodule of $(A \oplus M)_A$. Hence if $v = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, then vT is a minimal right ideal of T hence $lr(v) \subseteq Tv$.
Now to find $lr(v)$: Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in r(v)$ then

$\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ma_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives $a_1 \in r_A(m)$. Thus $r(v) = \begin{bmatrix} r_A(m) & 0 \\ M & B \end{bmatrix}$.

To find $lr(v)$, let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in lr(v)$ then $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} r_A(m) & 0 \\ M & B \end{bmatrix} = \begin{bmatrix} a_1 r_A(m) & 0 \\ m_1 r_A(m) + b_1 M & b_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives $a_1 \in l_A r_A(m)$, $b_1 = 0$, $m_1 \in l_M r_A(m)$. Therefore $lr(v) = \begin{bmatrix} l_A r_A(m) & 0 \\ l_M r_A(m) & 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $Tv = \begin{bmatrix} 0 & 0 \\ Bm & 0 \end{bmatrix}$. Since T is right mininjective $lr(v) \subseteq Tv$; that is $l_A r_A(m) = 0$, $l_M r_A(m) \subseteq Bm$.

Now it remains to show that B is right mininjective. From Proposition III.1 [9] it suffices to show that $l_B(K)$ is simple or zero for all maximal right ideals K of B. Let K be any maximal right ideal of B then $\tilde{K} = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}$ is a maximal right ideal of T. Since T is right mininjective $l_T(\tilde{K})$ is simple or zero. To find $l_T(\tilde{K})$:Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in l_T(\tilde{K})$, then $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} A & 0 \\ M & K \end{bmatrix} = \begin{bmatrix} a_1 A & 0 \\ m_1 A + b_1 M & b_1 K \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives $a_1 = 0$, $b_1 K = 0$ gives $b_1 \in l_B(K)$, $m_1 A = 0$ gives $m_1 = 0$, $b_1 M = 0$ gives $b_1 \in l_B(M)$. Therefore $l_T(\tilde{K}) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(M) \cap l_B(K) \end{bmatrix}$, so if $l_T(\tilde{K})$ is simple or zero implies that $l_B(M) \cap l_B(K)$ is minimal right ideal of B. Since $l_B(M)$ is essential left ideal and $l_B(K) \subseteq Soc(_B B)$ so $l_B(K) \cap l_B(M) = l_B(K)$, hence $l_B(K)$ is either simple or zero for all maximal ideals K of B showing that B is right mininjective.

(2) and (4) $\Rightarrow$ (1) We need only to prove that (2) and (4) imply right mininjectivity of B. The other statements in (1) follow as in proof of (2) and (3) $\Rightarrow$ (1). Let $b \in B$ with bB simple. Since $l_B(M)$ is an essential right ideal of B we get $bB \subseteq Soc(B_B) \subseteq l_B(M)$.

$u = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ then $uT = \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix}$ is minimal right ideal of T. By the right mininjectivity of T we get $lr(u) = Tu$. Now to find $lr(u)$: $u = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, to find $r(u)$; let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in r(u)$ then $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies that $bm_1 = 0$ and $bb_1 = 0$. Now $bm_1 = 0$ gives $m_1 \in M$ since $bM = 0$ and $bb_1 = 0$ gives $b_1 \in r_B(b)$, therefore $r(u) = \begin{bmatrix} A & 0 \\ M & r_B(b) \end{bmatrix}$.

Now to find $lr(u)$: Let $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \in lr(u)$ then $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} A & 0 \\ M & r_B(b) \end{bmatrix} = \begin{bmatrix} a_1 A & 0 \\ m_1 A + b_1 M & b_1 r_B(b) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives us $a_1 A = 0$; that is $a_1 = 0$, $b_1 r_B(b) = 0$; that is $b_1 \in l_B r_B(b)$ and $m_1 A + b_1 M = 0$ imply that $m_1 A = 0$ giving $m_1 = 0$ and $b_1 M = 0$ giving $b_1 \in l_B(M)$. Therefore $lr(u) = \begin{bmatrix} 0 & 0 \\ 0 & l_B(M) \cap l_B r_B(b) \end{bmatrix}$ and $Tu = \begin{bmatrix} 0 & 0 \\ 0 & Bb \end{bmatrix}$. Since $lr(u) = Tu$ we have $l_B(M) \cap l_B r_B(b) = Bb$. Also since $r_B(b)$ is maximal ideal of B, we have $J(B) \subseteq r_B(b)$ hence $l_B r_B(b) \subseteq l_B(J(B))$, by assumption $l_B(J(B)) = Soc(B_B)$ and so $l_B r_B(b) \subseteq Soc(B_B) \subseteq l_B(M)$. Hence $l_B(M) \cap l_B r_B(b) = l_B r_B(b) = Bb$ proving that B is right mininjective. □

Definition III.3. A ring R is called strong right mininjective if $l_R(I) = 0$ for every maximal right ideal I of R.

Since every simple right ideal of R is of the form aR with $r_R(a)$ maximal right ideal of R. $l_R r_R(a) = Ra$ if R is right mininjective. It follows that R is strong right mininjective if $l_R r_R(a) = 0$ implies that $Ra = 0$; that is $a = 0$. This means that R has no simple right ideals so Socle(R_R)=0. That is a ring R is strong right mininjective $\Leftrightarrow$ Socle(R_R) = 0. R is strong right mininjective $\Leftrightarrow$ the dual of every simple right R-module is zero.

Corollary III.1. *Let $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$. Then T is strong right mininjective if and only if A is strong right mininjective, $Soc(M_A) = 0$ and $Soc(L_B) = 0$ where $L = l_B(M)$.*

Proof. By Corollary II.1 $\text{soc}(T_T) = \begin{bmatrix} \text{soc}(A_A) & 0 \\ \text{soc}(M_A) & \text{soc}(L_B) \end{bmatrix}$, $L = l_B(M)$ then T is strong right mininjective if and only if soc(T_T)=0. □

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Lemma III.4. *$SocT_T$ is an essential right ideal of T if and only if the following are valid: (i) $SocA_A$ is essential in A_A.*
(ii) $SocM_A$ is essential in M_A.
(iii) If $l_B(M) \neq 0$, $Soc(l_B(M)_B)$ is essential in $l_B(M)_B$.

Proof. By Corollary II.1 we have $SocT_T = \begin{bmatrix} Soc(A_A) & 0 \\ Soc(M_A) & soc(L_B) \end{bmatrix}$ where $L_B = l_B(M)$. Assume that $Soc(T_T)$ is essential in T_T. For any $\alpha \neq 0$ in A and $m \neq 0$ in M we have $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} aA & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} T = \begin{bmatrix} 0 & 0 \\ mA & 0 \end{bmatrix}$. From $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} T \cap Soc(T_T) \neq 0$ and $\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} T \cap Soc(T_T) \neq 0$ we immediately see that $aA \cap Soc(A_A) \neq 0$ and $mA \cap Soc(M_A) \neq 0$. Also if $0 \neq b \in L_B$; we have $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} T \cap Soc(T_T) \neq 0$ and so we get $bB \cap Soc(L_B) \neq 0$. These yields (i),(ii) and (iii).

Conversely, assume (i),(ii)and (iii). Since any principal right ideal $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} T$ can be written as $\begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix}$. Because of Lemma II.1, to show that $SocT_T$ is essential in T_T it suffices to show that $\begin{bmatrix} a \\ m \end{bmatrix} A \cap SocT_T \neq 0$ and $\begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix} \cap SocT_T \neq 0$ whenever $(a, m) \neq (0, 0) \in A \oplus M$ or $b \neq 0$ in B. Conditions (i) and (ii) clearly imply $\begin{bmatrix} a \\ m \end{bmatrix} A \cap SocT_T \neq 0$ whenever $(a, m) \neq (0, 0)$ in $A \oplus M$. If $bM \neq 0$ we have $\begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix} \cap SocT_T \supseteq \begin{bmatrix} 0 & 0 \\ bM & 0 \end{bmatrix} \cap \begin{bmatrix} 0 & 0 \\ Soc(M_A) & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & 0 \\ bM \cap Soc(M_A) & 0 \end{bmatrix} \neq 0$. If on the other hand $bM = 0$ then $0 \neq b \in L_B$ and $\begin{bmatrix} 0 & 0 \\ 0 & bB \end{bmatrix} \cap Soc(T_T) \supseteq \begin{bmatrix} 0 & 0 \\ 0 & bB \cap Soc(L_B) \end{bmatrix} \neq 0$. □

Corollary III.2. *The ordinary triangular matrix ring $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ is strong right mininjective $\Leftrightarrow$ A is so.*

In this section we will see some properties passing over to triangular matrix ring. We will prove that certain properties like having n in the stable range, being a potent ring or being a clean ring all pass over to triangular matrix rings. Throughout this section T will denote the triangular matrix ring $\begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$. Let R be a ring and n an integer ≥ 1. Recall that a row $(a_1, a_2 ... a_n)$ of elements of R is called a right unimodular n row if there exists elements $\lambda_i \in R$ for $1 \leq i \leq n$ satisfying $\sum_{i=1}^{n} a_i \lambda_i = 1$. Let $U_r(n, R)$ denote the set of right unimodular n rows over R. Recall that n is said to be in the stable range of R if for any $(a_1, ..., a_{n+1}) \in U_r(n+1, R)$ we can find elements $c_i \in R$ for $1 \leq i \leq n$ satisfying the condition that
$(a_1 + a_{n+1}c_1, ..., a_n + a_{n+1}c_n) \in U_r(n, R)$.

Lemma III.5. *Let $t_i = \begin{bmatrix} a_i & O \\ m_i & b_i \end{bmatrix} \in T$ for $1 \leq i \leq n$. Then*
$(t_1, ..., t_n) \in U_r(n, T)$ if and only if $(a_1, ..., a_n) \in U_r(n, A)$ and
$(b_1, ..., b_n) \in U_r(n, B)$.

Proof. Assume that $(t_1, ..., t_n) \in U_r(n, T)$. Then there exists $s_i = \begin{bmatrix} \lambda_i & O \\ \omega_i & \mu_i \end{bmatrix} \in T$ for $1 \leq i \leq n$ satisfying $\sum_{i=1}^{n} t_i s_i = 1$ in T. This in particular yields $\sum_{i=1}^{n} a_i \lambda_i = 1$ in A and $\sum_{i=1}^{n} b_i \mu_i = 1$ in B. Hence $(a_1, ..., a_n) \in U_r(n, A)$ and $(b_1, ..., b_n) \in U_r(n, B)$.
conversely, assume that $(a_1, ..., a_n) \in U_r(n, A)$ and $(b_1, ..., b_n) \in U_r(n, B)$. To show that $(t_1, ..., t_n) \in U_r(n, T)$. let $\lambda_i \in A$ and $\mu_i \in B$ satisfy $\sum_{i=1}^{n} a_i \lambda_i = 1$ and $\sum_{i=1}^{n} b_i \mu_i = 1$. Now to find $s_i = \begin{bmatrix} \lambda_i & 0 \\ \omega_i & \mu_i \end{bmatrix}$ such that $\sum_{i=1}^{n} t_i s_i = 1$ in T. Let $v = \sum_{i=1}^{n} m_i \lambda_i$ define $\omega_i = -\mu_i v$, then $\sum_{i=1}^{n} b_i \omega_i = -\sum_{i=1}^{n} b_i(\mu_i v) = -\sum_{i=1}^{n} b_i \mu_i v = -v$ and define $s_i = \begin{bmatrix} \lambda_i & 0 \\ \omega_i & \mu_i \end{bmatrix}$ then simple checking shows that $\sum_{i=1}^{n} t_i s_i = 1$ in T. Hence $(t_1, ..., t_n) \in U_r(n, T)$. □

Theorem III.6. *n is in the stable range of T if and only if n is in the stable range of A and n is in the stable range of B.*

Proof. Assume n is in the stable range of T. Let $(a_1, ..., a_{n+1}) \in U_r(n+1, A)$ and $(b_1, ..., b_{n+1}) \in U_r(n+1, B)$. Let $t_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix} \in T$, since n is in the stable range of T there exists $q_i = \begin{bmatrix} c_i & 0 \\ u_i & d_i \end{bmatrix} \in T$ for $1 \leq i \leq n$ such that $(t_1 + t_{n+1}q_1, t_2 + t_{n+1}q_2, ..., t_n + t_{n+1}q_n) \in U_r(n, T)$. By Lemma III.5 $(a_1 + a_{n+1}c_1, ..., a_n + a_{n+1}c_n) \in U_r(n, A)$ and
$(b_1 + b_{n+1}d_1, ..., b_n + b_{n+1}d_n) \in U_r(n, B)$, hence n is in the stable range of both A and B.
Conversely, assume that n is in the stable range of both A and B. Let
$t_i = \begin{bmatrix} a_i & 0 \\ m_i & b_i \end{bmatrix} \in T$ for $1 \leq i \leq n+1$, satisfy $(t_1, ..., t_{n+1}) \in U_r(n+1, T)$; that is there exists $s_i = \begin{bmatrix} \lambda_i & 0 \\ \omega_i & \mu_i \end{bmatrix}$ for $1 \leq i \leq n+1$ in T such that $\sum_{i=1}^{n+1} t_i s_i = 1$ in T.

From Lemma III.5. we see that $\sum_{i=1}^{n+1} a_i \lambda_i = 1$ in A and $\sum_{i=1}^{n+1} b_i \mu_i = 1$ in B. That is $(a_1, ..., a_{n+1}) \in U_r(n+1, A)$ and $(b_1, ..., b_{n+1}) \in U_r(n+1, B)$. Since n is in the stable range of both A, B we get elements $c_i \in A,\ d_i \in B$ for
$1 \leq i \leq n$ satisfying the condition that $(a_1 + a_{n+1}c_1, ..., a_n + a_{n+1}c_n)$ is in $U_r(n, A)$ and $(b_1 + b_{n+1}d_1, ..., b_n + b_{n+1}d_n)$ is in $U_r(n, B)$. Let $q_i = \begin{bmatrix} c_i & 0 \\ 0 & d_i \end{bmatrix} \in T$ for $1 \leq i \leq n$, then $t_i + t_{n+1}q_i = \begin{bmatrix} a_i + a_{n+1}c_i & 0 \\ m_i + m_{n+1}c_i & b_i + b_{n+1}d_i \end{bmatrix}$. From Lemma III.5 we see that $(t_1 + t_{n+1}q_1, ..., t_n + t_{n+1}q_n) \in U_r(n, T)$. Hence n is in the stable range of T.

□

A ring R is said to be Clean [10] if every element in R is the sum of a unit and an idempotent. If $\theta : R \to S$ is a surjective ring homomorphism and R is clean, then so is S.

Proposition III.2. *T is clean if and only if A and B are clean.*

Proof. Since A and B are factor rings of T, to prove the proposition we have only to show that if A and B are clean then so is T. Let $u = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in T$.
Then $a = e + \alpha, b = f + \beta$ where, $e^2 = e \in A, f^2 = f \in B, \alpha$ is a unit in A and β is a unit in B. Then $u = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ m & \beta \end{bmatrix}$ with $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ an idempotent and $\begin{bmatrix} \alpha & 0 \\ m & \beta \end{bmatrix}$ a unit in T.

□

Recall that a ring R is said to be potent if idempotents mod $J(R)$ can be lifted and every right (equivalently left) ideal L of R with $L \subsetneq J(R)$ contains a non zero idempotent.

Theorem III.7. *T is a potent ring if and only if A and B are potent rings.*

Proof. From Corollaries II.1, II.2 we have $J(T) = \begin{bmatrix} J(A) & 0 \\ M & J(B) \end{bmatrix}$ and idempotents mod $J(T)$ can be lifted in T if and only if idempotents mod $J(A)$ can be lifted in A and idempotents mod $J(B)$ can be lifted in B. Assume T is potent. Let I be a right ideal of A not contained in $J(A)$. Then $\tilde{I} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is a right ideal of T not contained in $J(T)$. Hence there exists an element $0 \neq e \in I$ with $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ in T, equivalently $e^2 = e$ in A. Let K be a right ideal of B with $K \nsubseteq J(B)$. Then $\tilde{K} = \begin{bmatrix} 0 & 0 \\ KM & K \end{bmatrix}$ is a right ideal of T with $\tilde{K} \nsubseteq J(T)$. Hence there exists an element $\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix} \in \tilde{K}$ with $\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix}\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix}$. But since $\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix}\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ fm & f^2 \end{bmatrix}$, we get $f^2 = f$ and $fm = m$. From $\begin{bmatrix} 0 & 0 \\ m & f \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we see that either $f \neq 0$ or $m \neq 0$. From $fm = m$ we conclude that $f \neq 0$. Thus $0 \neq f \in K$ and $f^2 = f$ in B. This proves implication T potent $\Rightarrow A$ and B are potent.

Now assume that A and B are potent rings. Let $\tilde{I}$ be a right ideal of T with $\tilde{I} \nsubseteq J(T)$. Then there exists an element $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in \tilde{I}$ with

$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \notin J(T)$. This means that either $a \notin J(A)$ or $b \notin J(B)$. To show that $\tilde{I}$ contains a nonzero idempotent it suffices to show that $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} T = \begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix}$ contains a non zero idempotent. If $b \notin J(B)$ there exists a non zero idempotent $f \in bB$ and $\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$ is a nonzero idempotent in uT where $u = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix}$. Suppose $a \notin J(A)$. Then there exists a nonzero idempotent e in A of the form $a\lambda$ for some $\lambda \in A$. Then $a\lambda e = e.e = e \neq 0$. The element $\begin{bmatrix} e & 0 \\ m\lambda e & 0 \end{bmatrix} = \begin{bmatrix} a\lambda e & 0 \\ m\lambda e & 0 \end{bmatrix}$ is in $\begin{bmatrix} a \\ m \end{bmatrix} A$, hence $\begin{bmatrix} e & 0 \\ m\lambda e & 0 \end{bmatrix} \in uT$. Also $\begin{bmatrix} e & 0 \\ m\lambda e & 0 \end{bmatrix}\begin{bmatrix} e & 0 \\ m\lambda e & 0 \end{bmatrix} = \begin{bmatrix} e & 0 \\ m\lambda e & 0 \end{bmatrix}$. Thus $\begin{bmatrix} e & 0 \\ m\lambda e & 0 \end{bmatrix}$ is a nonzero idempotent in uT. This proves that T is potent. □

Definition III.4. An element u of a ring R is said to be right repetitive if for each finitely generated right ideal I of R, the right ideal $\sum_{n\geq 0} u^n I$ is finitely generated; equivalently there exists an integer $k \geq 1$ (depending on u and I) satisfying $u^k I \subseteq \sum_{n=0}^{k-1} u^n I$ [6].

R itself is said to be right repetitive if every element in R is right repetitive. The importance of right repetitive rings arises from the fact that over such rings all cyclic right modules are hopfian. Recall that a module N is hopfian if every surjective endomorphism $f : N \to N$ is automatically an isomorphism.

Lemma III.6. *Let $\theta : R \to S$ be a surjective ring homomorphism. If R is right repetitive so is S.*

Proof. Let K be a finitely generated right ideal of S and $v \in S$. By lifting a finite set of generators of S to R we get a finitely generated right ideal I of R satisfying $\theta(I) = K$. Let $u \in R$ satisfying $\theta(u) = v$. Then since R is right repetitive and I is finitely generated there exists an integer $k \geq 1$ with $u^k I \subseteq \sum_{n=0}^{k-1} u^n I$. From $\theta(u^n I) = v^n K$ we see that $v^k K \subseteq \sum_{n=0}^{k-1} v^n K$. Hence S is right repetitive. □

Theorem III.8. *Let $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$.*
(1) If T is right repetitive so are A and B.
(2) If $M = 0$ or M_A is simple then the right repetitiveness of A and B implies the right repetitiveness of T.

Proof. (1) If $a \in A$ and I be finitely generated right ideal of A then $u = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in T$ and $\tilde{I} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is finitely generated right ideal of T. Since T is right repetitive there exists $K \geq 1$ such that $u^k \tilde{I} \subseteq \sum_{n=0}^{k-1} u^n \tilde{I}$. We have $u = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ $u^k \tilde{I} = \begin{bmatrix} a^k I & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} a^k I & 0 \\ 0 & 0 \end{bmatrix} \subseteq \sum_{n=0}^{k-1} \begin{bmatrix} a^n I & 0 \\ 0 & 0 \end{bmatrix}$ which implies that $\begin{bmatrix} a^k I & 0 \\ 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} \sum_{n=0}^{k-1} a^n I & 0 \\ M & B \end{bmatrix}$. Therefore $a^k I \subseteq \sum_{n=0}^{k-1} a^n I$; that is A is right repetitive.

Now let $b \in B$, J be any finitely right ideal of B then $v = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in T$ and $\tilde{J} = \begin{bmatrix} 0 & 0 \\ JM & J \end{bmatrix}$ is finitely generated right ideal of T. Since T is right repetitive there exists $k \geq 1$ such that $v^k \tilde{J} \subseteq \sum_{n=0}^{k-1} v^n \tilde{J}$; that is $\begin{bmatrix} 0 & 0 \\ 0 & b^k \end{bmatrix}\begin{bmatrix} 0 & 0 \\ JM & J \end{bmatrix} \subseteq \sum_{n=0}^{k-1} \begin{bmatrix} 0 & 0 \\ 0 & b^n \end{bmatrix}\begin{bmatrix} 0 & 0 \\ JM & J \end{bmatrix}$ which implies that $\begin{bmatrix} 0 & 0 \\ b^k JM & b^k J \end{bmatrix} \subseteq \sum_{n=0}^{k-1} \begin{bmatrix} 0 & 0 \\ b^n JM & b^n J \end{bmatrix}$; that is $b^k J \subseteq \sum_{n=0}^{k-1} b^n J$. Therefore B is right repetitive.

(2) Assume that either $M = 0$ or M_A is simple and then A and B are right repetitive. To show that T is right repetitive we need to show that any element of T acts repetitively on any principal right ideal of T. Let $\tilde{I} = \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} T$ and $u = \begin{bmatrix} \alpha & 0 \\ y & \beta \end{bmatrix}$ be any element of T. We have $\tilde{I} = \begin{bmatrix} a \\ m \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ bM & bB \end{bmatrix}$ from Lemma II.1. Also $u = \begin{bmatrix} \alpha & 0 \\ y & \beta \end{bmatrix}$, $u^2 = \begin{bmatrix} \alpha & 0 \\ y & \beta \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ y & \beta \end{bmatrix}$

$$= \begin{bmatrix} \alpha^2 & 0 \\ y\alpha + \beta y & \beta^2 \end{bmatrix}$$

$$u^3 = \begin{bmatrix} \alpha^3 & 0 \\ y\alpha^2 + \beta y\alpha + \beta^2 y & \beta^3 \end{bmatrix}$$

$$u^4 = \begin{bmatrix} \alpha^4 & 0 \\ y\alpha^3 + \beta y\alpha^2 + \beta^2 y\alpha + \beta^3 y & \beta^4 \end{bmatrix}$$

so in general $u^k =$

$\left[\begin{array}{cc} \alpha^k & 0 \\ y\alpha^{k-1}+\beta y\alpha^{k-2}+..+\beta^{k-1}y & \beta^k \end{array}\right]$ and

$u^k\tilde{I} = u^k\left[\begin{array}{cc} a & 0 \\ m & b \end{array}\right]T =$

$\left[\begin{array}{cc} \alpha^k a & 0 \\ y\alpha^{k-1}a+\beta y\alpha^{k-2}a+...+\beta^{k-1}ya+\beta^k m & \beta^k b \end{array}\right]T$

$= \left[\begin{array}{c} \alpha^k a \\ y\alpha^{k-1}a+\beta y\alpha^{k-2}a+...+\beta^{k-1}ya+\beta^k m \end{array}\right]A$

$+ \left[\begin{array}{cc} 0 & 0 \\ \beta^k bM & \beta^k bB \end{array}\right]$. Thus a typical element of $u^k\tilde{I}$ is of the following form :

$\left[\begin{array}{cc} \alpha^k a\lambda & 0 \\ (y\alpha^{k-1}a+\beta y\alpha^{k-2}a+...+\beta^{k-1}ya+\beta^k m)\lambda+\beta^k bx & \beta^k b\mu \end{array}\right]$,

where $\lambda \in A,\ x \in M$ and $mu \in B$. Thus u acts repetitively on $\tilde{I} \Leftrightarrow$ for some integer $k \geq 2$ the following three conditions are valid:

(1) $\alpha^{k+1}aA \subseteq aA + \alpha aA + ... + \alpha^k aA$

(2) $\beta^{k+1}bB \subseteq bB + \beta bB + ... + \beta^k bB$

(3) $(y\alpha^k a + \beta y\alpha^{k-1}a + ... + \beta^k ya + \beta^{k+1}m)A + \beta^{k+1}bM \subseteq mA + bM + (ya+\beta m)A + \beta bM + \sum_{i=1}^{k-1}(y\alpha^i a + \beta y\alpha^{i-1}a + ... + \beta^i ya + \beta^{i+1}m)A + \sum_{i=1}^{k-1}\beta^{i+1}bM.$

$\{$Since $u^{k+1}\tilde{I} \subseteq \sum_{i=0}^{k} u^i\tilde{I} = \tilde{I} + u\tilde{I} + ... + u^k\tilde{I}$

$\tilde{I} = \left[\begin{array}{c} a \\ m \end{array}\right]A + \left[\begin{array}{cc} 0 & 0 \\ bM & bB \end{array}\right]$

$u\tilde{I} = \left[\begin{array}{c} \alpha a \\ ya+\beta m \end{array}\right]A + \left[\begin{array}{cc} 0 & 0 \\ \beta bM & \beta bB \end{array}\right]$

$u^k\tilde{I} =$
$\left[\begin{array}{c} \alpha^k a \\ y\alpha^{k-1}a+\beta y\alpha^{k-2}a+...+\beta^{k-1}ya+\beta^k m \end{array}\right]A+$
$\left[\begin{array}{cc} 0 & 0 \\ \beta^k bM & \beta^k bB \end{array}\right]$

$u^{k+1}\tilde{I} =$
$\left[\begin{array}{c} \alpha^{k+1} a \\ y\alpha^k a+\beta y\alpha^{k-1}a+...+\beta^k ya+\beta^{k+1} m \end{array}\right]A+$
$\left[\begin{array}{cc} 0 & 0 \\ \beta^{k+1} bM & \beta^{k+1} bB \end{array}\right]\}$ Now (1) is equivalent to saying that $\alpha \in A$ acts repetitively on aA, (2) is equivalent to saying that $\beta \in B$ acts repetitively on bB, (3) could be written as (3)'

$(y\alpha^k a + \beta y\alpha^{k-1}a + ... + \beta^k ya + \beta^{k+1}m)A + \beta^{k+1}bM \subseteq mA + (ya+\beta m)A + \sum_{i=1}^{k-1}(y\alpha^i a + \beta y\alpha^{i-1}a + ... + \beta^i ya + \beta^{i+1}m)A + bM + \beta bM +$

$\sum_{i=1}^{k-1}\beta^{i+1}bM$

When $M = 0$ this is equivalent to show $0 \subseteq 0$ which is clearly true.

If $m \neq 0$ or $bM \neq 0$ the right hand side of (3)' is whole of M when M_A is simple. Hence (3) is valid in this case. Let $m = 0$ and $bM = 0$, If one of $y\alpha^i a + \beta y\alpha^{i-1}a + ... + \beta^i ya$ is not 0, say for $i = l$ then (3)' is valid with $k = l+1$. If $y\alpha^i a + \beta y\alpha^{i-1}a + ... + \beta^i ya = 0$ for all $i \geq 1$ then the L.H.S. of (3)' is zero, hence (3)' is valid. This completes the proof of the theorem. □

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